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Non-Newtonian Fluid Flow on a Flat Plate Part 1: Boundary Layer

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A modified power-law viscosity for non-Newtonian fluids based on actual measurements is proposed. This realistic model allows removal of the singularities at the leading edge of a flat-plate boundary layer for either shear-thinning or shear-thickening fluids. Under this condition, the boundary-layer equations can be solved numerically by simple finite difference methods that march downstream from the leading edge, as is usually done for Newtonian fluids. Numerical results are presented for the case of a shear-thinning fluid; applying the model to a shear-thickening fluid is straightforward. The effects of this new variable viscosity are explicitly demonstrated by comparing plots of isolines of viscosity and shear rate, the velocity distribution, and the wall shear stress for non-Newtonian and Newtonian fluids.

Nomenclature

C	=	constant
C_f	=	dimensionless wall shear stress
D	=	nondimensional fluid viscosity
l	=	reference length of the plate
n	=	power-law index
Re	=	Reynolds number, $U_0 l/\nu_1$
U_0	=	freestream velocity
(U, V)	=	dimensionless fluid velocities in the (ξ, η) directions
$(\bar{u}, \bar{v})$	=	dimensional fluid velocities in the $(\bar{x}, \bar{y})$ directions
γ	=	fluid shear rate
η	=	pseudosimilarity variable
ν	=	viscosity of the non-Newtonian fluid
ξ	=	axial direction along the plate
ρ	=	fluid density

I. Introduction

THE major motivation for studying the dynamics of non-Newtonian fluids is their widespread use in applications. A recent lecture on non-Newtonian fluids was given by Hinch [1]. It appears that Acrivos [2], a frequently cited paper, was the first to consider boundary-layer flows for power-law non-Newtonian fluids. Since then, a large number of related papers have been published; for additional references, see [3–5].

Power-law correlations for shear-thinning or shear-thickening fluids can lead to zero or infinite viscosity when the shear rate becomes zero or infinite. This usually occurs at the leading edge of boundary layers or along the outer edge of boundary layers in which the boundary layer matches the outer inviscid flow. Thus, power-law correlations introduce nonremovable singularities into boundary-layer formulations. The reason why such singularities cannot be removed is due to the fact that power-law correlations are physically unrealistic in the limits of zero and infinite shear rates.

Two widespread mistakes appear continuously in papers studying boundary layers of power-law non-Newtonian fluids. The first is that few authors recognize that a length scale is introduced into boundary-layer formulations; consequently, boundary-layer problems with power-law non-Newtonian fluids cannot have simple self-similar solutions. It is nevertheless a common practice to ignore, without justification, the dependence of boundary-layer solutions on the streamwise coordinate. We will demonstrate in this paper that such a self-similar solution is actually only valid at the leading edge of the boundary layer. The similarity solution is a natural upstream condition that is needed to integrate boundary-layer equations along the streamwise direction from the leading edge. The mathematical structure of the boundary layer is similar to that of the mixed-convection boundary layer on a vertical heated plate [6].

It is not difficult to check whether a boundary-layer flow has a self-similar solution. One can always keep the dependence of the solution on the streamwise coordinate in the formulation of a similarity transformation or in a system of parabolic coordinates. If the numerical solution obtained by integrating downstream from the leading edge does not change from one streamwise location to another, then the boundary-layer flow has a self-similar solution. This procedure is used to demonstrate that a boundary-layer flow of power-law non-Newtonian fluids does not have a self-similar solution. The second error is less serious than the first one. Some authors have recognized that self-similar solutions do not exist and have also realized that the singularity at the leading edge is not removable. Thus, they started their integration slightly downstream from the leading edge, which is known as a fault-starting process [3]. How far upstream errors propagate downstream is an unanswered problem.

We propose a modified power-law correlation to remove the two physically unrealistic limits of zero and infinite viscosities from traditional power-law correlations [1]. The constants in the proposed model are fixed with available measurements and are described in detail. After removing these limits, the boundary-layer equations can be analyzed by methods established for Newtonian fluid with a physically realistic variable viscosity. The physical model studied in the paper is a boundary layer on a flat plate in a uniform outer flow. For Newtonian fluids, the solution is the well-known Blasius solution. The numerical solution for a shear-thinning fluid shows that the effect of a non-Newtonian fluid can last quite far downstream.

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The sensitivity of the proposed correlation to the power-law index will be reported in a separate paper because it contains a large amount of computational results.

II. Formulation of the Problem

We study two-dimensional laminar boundary layer of non-Newtonian fluid on a semi-infinite flat plate. The viscosity depends on the shear rate and is correlated by a modified power law. We concentrate our study on a shear-thinning fluid. The considered configuration is shown in Fig. 1.

The boundary-layer equations are

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{1}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\partial}{\partial \bar{y}}\left(\nu \frac{\partial \bar{u}}{\partial \bar{y}}\right) \tag{2}$$

where $(\bar{u}, \bar{v})$ are velocity components along the $(\bar{x}, \bar{y})$ axes, and ρ is the density.

The viscosity is correlated by a modified power law, which is

$$\nu = \frac{K}{\rho}\left|\frac{\partial \bar{u}}{\partial \bar{y}}\right|^{n-1} \quad \text{for } \bar{\gamma}_1 \leq \left|\frac{\partial \bar{u}}{\partial \bar{y}}\right| \leq \bar{\gamma}_2 \tag{3}$$

The constants $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are two threshold velocity gradients (dimensionless shear), and K is a dimensional constant for which the dimension depends on the power-law index n. The values of these constants can be determined by measurements. Outside the preceding range, viscosity is approximately constant, and its value can be fixed with data. The associated boundary conditions are

$$\bar{u} = \bar{v} = 0 \quad \text{at } \bar{y} = 0 \tag{4}$$

$$\bar{u} \to U_0 \quad \text{as } \bar{y} \to \infty \tag{5}$$

where U_0 is the freestream velocity.

We now introduce the following nondimensional transformations:

$$x = \frac{\bar{x}}{l}, \qquad y = \frac{\bar{y}}{l} Re^{1/2}, \qquad u = \frac{\bar{u}}{U_0}$$
$$v = \frac{\bar{v}}{U_0} Re^{1/2}, \qquad D = \frac{\nu}{\nu_1} \tag{6}$$

where ν_1 is a reference viscosity and l is the length scale, which will be subsequently described in detail. Substituting variables from Eq. (6) into Eqs. (1–3) leads to the following nondimensional equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{7}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[D\frac{\partial u}{\partial y}\right] \tag{8}$$

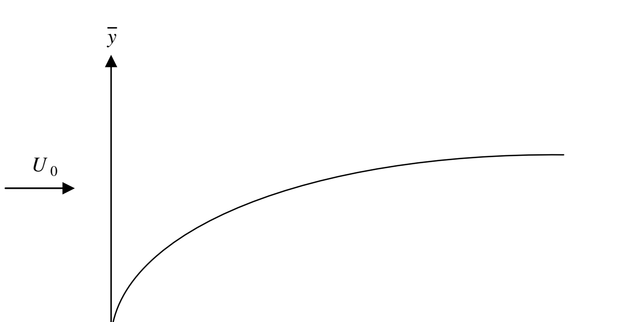

Fig. 1 Coordinates.

$$D = \frac{K}{\rho \nu_1}\left(\frac{U_0^3}{l\nu_1}\right)^{\frac{n-1}{2}}\left|\frac{\partial u}{\partial y}\right|^{n-1} = C\left|\frac{\partial u}{\partial y}\right|^{n-1} \tag{9}$$

where Re is the Reynolds number, and

$$D = \begin{cases} 1, & \gamma \leq \gamma_1 \\ C|\gamma|^{n-1}, & \gamma_1 \leq \gamma \leq \gamma_2 \\ 0.5, & \gamma \geq \gamma_2 \end{cases} \tag{10}$$

where $\gamma = \partial u/\partial y$.

For oil at about 68°F, the data determine the correlation shown in Fig. 2, in which $C = 0.891$ and $n = 0.95$. The length scale introduced by the power-law correlation is

$$l = C^{\frac{2}{1-n}}\left[\left(\frac{K}{\rho}\right)^2 \frac{1}{\upsilon_1^{n+1}}\right]^{\frac{1}{n-1}} U_0^3 \tag{11}$$

Because of this length scale, the current problem does not have a self-similar solution. The boundary conditions (4) and (5) become

$$u = v = 0 \quad \text{at } y = 0 \tag{12a}$$

$$u \to 1 \quad \text{as } y \to \infty \tag{12b}$$

The usual parabolic coordinates are used to remove the singularity at the leading edge:

$$\xi = x, \qquad \eta = \frac{y}{(2x)^{1/2}}, \qquad U = u, \qquad V = (2x)^{1/2} v \tag{13}$$

Consequently, Eqs. (13–15) become

$$(2\xi)\frac{\partial U}{\partial \xi} - \eta\frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \eta} = 0 \tag{14}$$

$$(2\xi)\frac{\partial U}{\partial \xi} + (V - \eta U)\frac{\partial U}{\partial \eta} = \frac{\partial}{\partial \eta}\left[D\frac{\partial U}{\partial \eta}\right] \tag{15}$$

and

$$\gamma = (2\xi)^{-1/2}\frac{\partial U}{\partial \eta} \tag{16}$$

Equations (14) and (15) can be easily solved by marching downstream, with the upstream condition satisfying the following ordinary differential equations:

$$-\eta\frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \eta} = 0 \tag{17}$$

$$(V - \eta U)\frac{\partial U}{\partial \eta} = \frac{\partial}{\partial \eta}\left[\frac{1}{2}\frac{\partial U}{\partial \eta}\right] \tag{18}$$

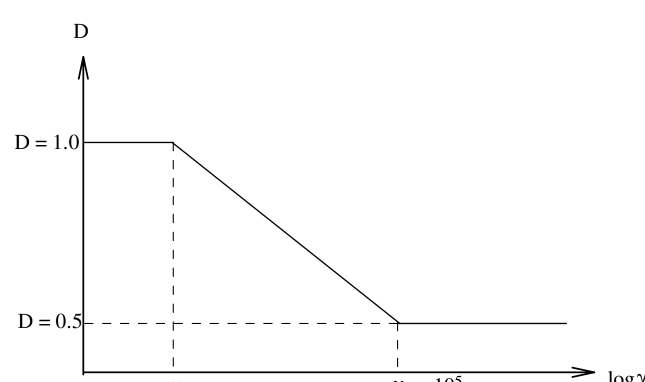

Fig. 2 Modified power-law correlation.

which are the limits of Eqs. (14) and (15) as $\xi \to 0$. The corresponding boundary conditions are

$$U = V = 0 \quad \text{at } \eta = 0 \tag{19a}$$

$$U \to 1 \quad \text{as } \eta \to \infty \tag{19b}$$

Equations (14) and (15) are discretized by a central-difference scheme for the diffusion term and by a backward-difference scheme for the convection terms; hence, the truncation errors are $\mathcal{O}(\Delta\xi)$. The computation is started from $\xi = 0$ and then marches to the downstream region $\xi = 100$. After several test runs, convergent results are obtained by using $\Delta\xi = 0.000002$ and $\Delta\eta = 0.002$ near the leading edge (say, ξ0.0–0.001). Downstream from $\xi = 0.001$, $\Delta\xi$ is gradually increased to 0.005.

III. Results and Discussion

The numerical results presented clearly demonstrate that singularities experienced for power-law correlation were successfully removed. Because the shear stress near the leading edge is inversely proportional to $\sqrt{2x}$, $D = 0.5$ at the leading edge for the modified power law. The velocity distributions as a function of η at selected ξ locations in Fig. 3 show that the variable-viscosity effect decreases downstream as the shear stresses decrease. The plot of isoviscosity lines are presented in Fig. 4a. The value of viscosity increases downstream, whereas the shear rate increases. The isoviscosity lines for small ξ are given separately, for clarity. The continuous variation of viscosity is shown in Fig. 4b. Even though we only plot the results up to $\xi = 10$, we have computed to $\xi \approx 5000$, for which the viscosity is still not uniform along the normal direction of the flat plate.

The distribution of D is illustrated in Fig. 5 for the different ξ positions as a function of η. Note that the value of D increases in the region of $\eta \leq 3.3$ and abruptly approaches the value of unity: the value of D at $\eta \approx 3$. This is due to the sudden change of the modified power-law correlation at γ_1. This abrupt change can be removed, but will further complicate the variable-viscosity correlation. It is also worthy to note that the fast change of solution near the leading edge is due to the fact that the thickness of the boundary layer is zero at the leading edge; consequently, the stress is infinite. The value of the shear stress drops very fast initially, when the thickness of the boundary layer grows downstream from the leading edge. This is expected and is not due to singularity, because it has already been removed.

The plot of iso-γ lines in Fig. 6 and the distribution of γ for the selected ξ position in Fig. 7 clearly show that the value of shear stresses is inversely proportional to $\sqrt{2\xi}$. This means that the maximum shear rate γ (and hence the minimum viscosity of the non-Newtonian fluid) is found at the leading edge of the plate.

The most interesting result is the variation of the wall shear stress,

$$C_f(2\xi)^{1/2} = \left[D\frac{\partial U}{\partial \eta}\right]_{\eta=0}$$

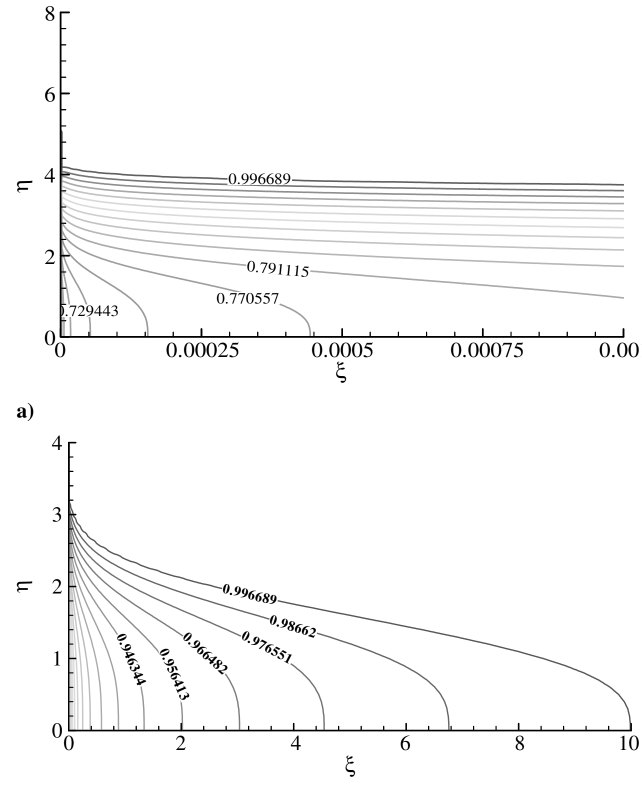

Fig. 4 Plots of isoviscosity D for a) $\xi = 0.0$–0.001 and b) $\xi = 0.0$–10.0.

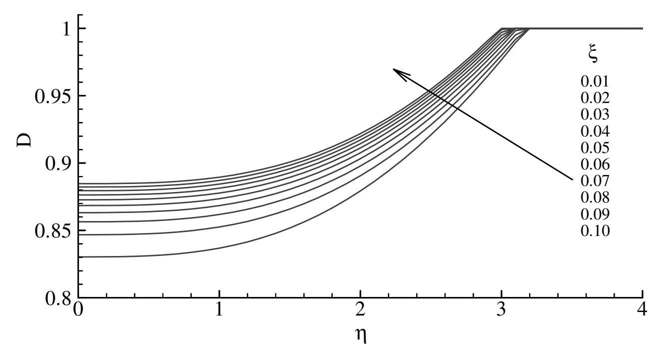

Fig. 5 Distribution of D at selected ξ.

which is calculated by a second-order finite difference scheme and is plotted in Fig. 8. The minimum wall shear $C_f(2\xi)^{1/2} \approx 0.33206$ occurs at the leading edge of the plate. Downstream of this region, the increase of D overwhelms the drop in γ; this leads to an increase of the wall shear stress to approximately 0.476 at $\xi = 9.96$. Then the wall shear stress gradually decreases and asymptotically approaches

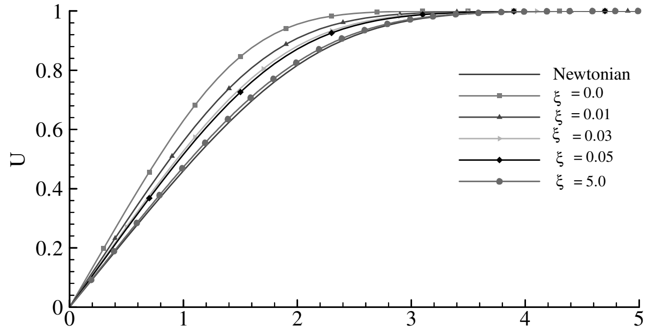

Fig. 3 Velocity distributions at various ξ.

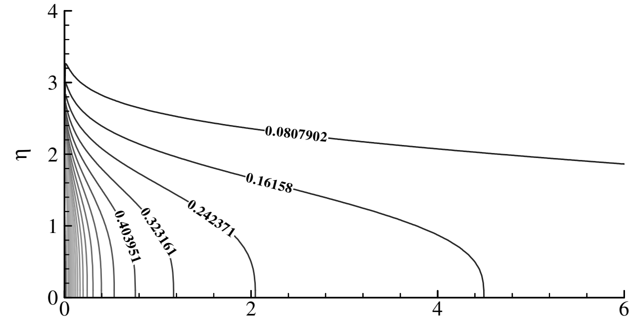

Fig. 6 Plot of iso-γ lines.

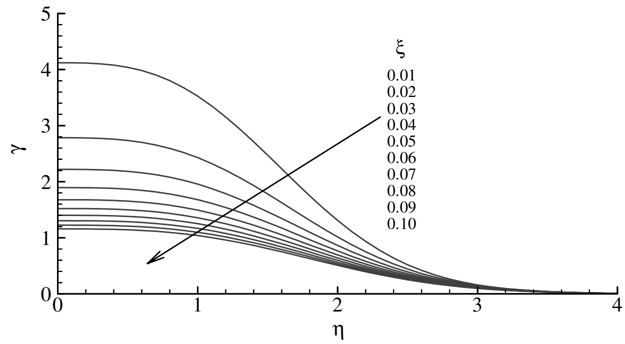

Fig. 7 Distribution of γ at selected ξ.

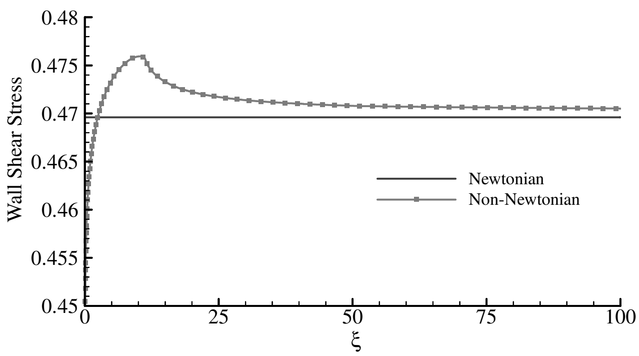

Fig. 8 Comparison of wall shear distributions.

the value for a Newtonian fluid. In a run up to $\xi \approx 5000$, the difference of wall shear stresses of non-Newtonian and Newtonian fluids becomes extremely small, but still not identical. This is a result of the inherent nonlinearity associated with the non-Newtonian fluids.

IV. Conclusions

The proposed modified power-law correlation matches well with the measurement of viscosities of non-Newtonian fluids; consequently, it is a physically realistic model. This model can be straightforwardly incorporated into boundary-layer equations for investigating boundary-layer flows and associated heat transfer problems of certain non-Newtonian fluids. The problems associated with the nonremovable singularity introduced by the traditional power-law correlations do not exist for the modified power-law correlation proposed in this paper.

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